**UNIVERSITÉ DU LUXEMBOURG**
**ANALYSE 1**
**2015-2016**

EXERCISE SHEET 7

**7.1.** For the following functions $f(x)$, determine the smallest real number $\alpha$ such that $\lim_{x\to 0^+} \frac{f(x)}{x^\alpha} \neq 0$. For such $\alpha$ compute the value of the limit.

**7.1.1.** $f(x) = \sin^3(x)$

**7.1.2.** $f(x) = \cos(x) - e^x$

**7.1.3.** $f(x) = \frac{1}{\sin(x)} - \frac{1}{x}$

**7.1.4.** $f(x) = \sin(\sinh(x)) - \sinh(\sin(x))$

**7.2.** Compute the limit of the following sequences according to the initial value $x_0 \in \mathbb{R}$:

**7.2.1.** $x_{n+1} = \frac{x_n}{1+x_n^2}$;

**7.2.2.** $x_{n+1} = x_n^2 - 6$

**7.3.** Consider the following sequence defined by recursion

$$\begin{cases} x_{n+1} = x_n\left(x_n + \frac{1}{n}\right) \\ x_1 = \alpha \geq 0 \end{cases}$$

Prove that there exists $\alpha_0 > 0$ such that

- if $\alpha > \alpha_0$ then $x_n \to +\infty$
- if $0 \leq \alpha < \alpha_0$ then $x_n \to 0$
- if $\alpha = \alpha_0$ then $x_n \to 1$ .

**7.4.** Consider the following sequence defined by recursion

$$\begin{cases} x_{n+1} = \frac{n}{n+1} x_n^2 \\ x_1 = \alpha \geq 0 \end{cases}$$

Prove that there exists $\alpha_0 > 0$ such that

- if $\alpha > \alpha_0$ then $x_n \to +\infty$
- if $0 \leq \alpha < \alpha_0$ then $x_n \to 0$
- if $\alpha = \alpha_0$ then $x_n \to 1$ .